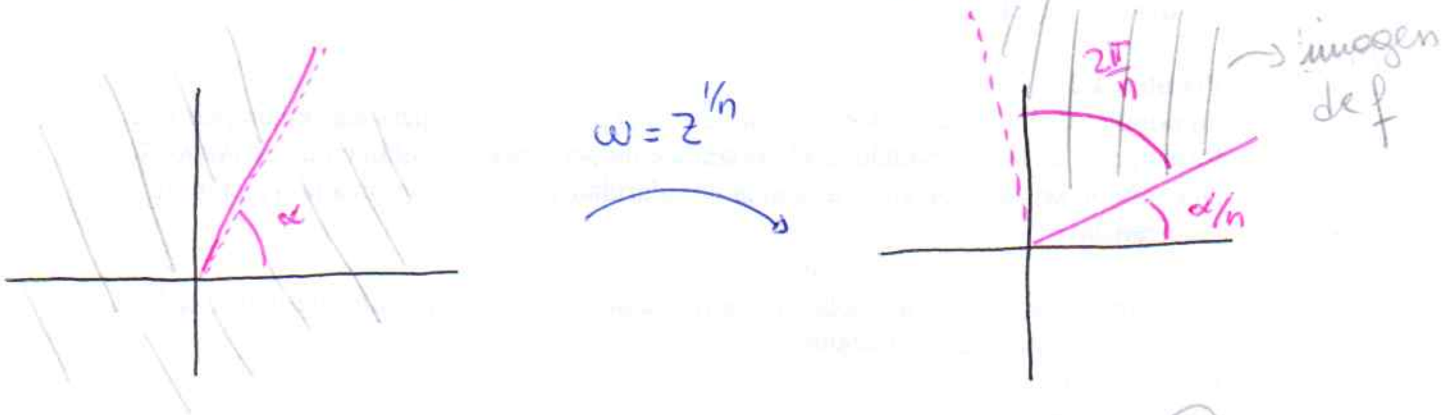
FUNCIÓN RAIZ N-ESIMA

$$\boxed{f(z) = z^{1/n} = |z|^{1/n} e^{i\frac{\arg z}{n}}}$$ → qué argumento?!

↓

determinamos UN argumento para cada $z \neq 0$

$$\alpha \leq \arg z < \alpha + 2\pi$$



$$\alpha \leq \arg z < \alpha + 2\pi \longrightarrow \frac{\alpha}{n} \leq \arg w < \frac{\alpha}{n} + \frac{2\pi}{n}$$

f es continua en $\mathbb{C} - \underbrace{\{z \in \mathbb{C} : \arg z = \alpha\}}_{\text{corte de rama}}$

f holomorfa en $\mathbb{C} - \{z \in \mathbb{C} : \arg z = \alpha\}$

$$f(z) = r^{1/n} e^{i\frac{\theta}{n}} \quad \text{con} \quad \alpha < \theta = \arg\alpha < \alpha + 2\pi$$

$$\left.\begin{array}{l} U(r,\theta) = r^{1/n} \cos\left(\frac{\theta}{n}\right) \\ V(r,\theta) = r^{1/n} \operatorname{sen}\left(\frac{\theta}{n}\right) \end{array}\right\} \text{ dif en } r > 0, \ \alpha < \theta < \alpha + 2\pi$$

$$U'_r(r,\theta) = \frac{1}{n} r^{1/n - 1} \cos\left(\frac{\theta}{n}\right) \qquad V'_\theta(r,\theta) = \frac{1}{n} r^{1/n} \cos\left(\frac{\theta}{n}\right)$$

$$U'_\theta(r,\theta) = \frac{-1}{n} r^{1/n} \operatorname{sen}\left(\frac{\theta}{n}\right) \qquad V'_r(r,\theta) = \frac{1}{n} r^{1/n - 1} \operatorname{sen}\left(\frac{\theta}{n}\right)$$

se verifica C-R.

C-R
$r . U'_r = V'_\theta$
$r V'_r = -U'_\theta$

$$f'(z) = e^{-i\theta}(U'_r + i V'_r)$$

$$f'(z) = e^{-i\theta}\left(\frac{1}{n}\frac{r^{1/n}}{r} \cos\left(\frac{\theta}{n}\right) + i \frac{1}{n}\frac{r^{1/n}}{r} \operatorname{sen}\left(\frac{\theta}{n}\right)\right) = \frac{1}{n} e^{-i\theta} . r^{1/n - 1} e^{i\frac{\theta}{n}}$$

$$= \frac{1}{e^{i\theta} r} \frac{1}{n} r^{1/n}\left(\cos\left(\frac{\theta}{n}\right) + i \operatorname{sen}\left(\frac{\theta}{n}\right)\right) = \frac{1}{n}\frac{z^{1/n}}{z} = \frac{1}{n} z^{1/n - 1}$$

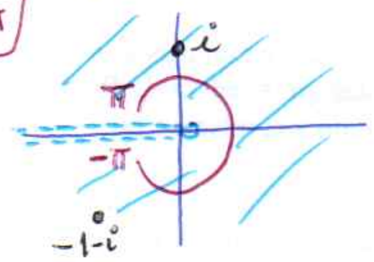
FUNCIÓN RAIZ CÚBICA

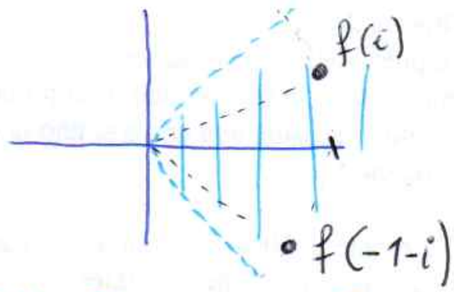
$$f(z) = z^{1/3} = |z|^{1/3}\, e^{i\frac{\arg z}{3}}$$

$$= \sqrt[3]{r}\, e^{i\frac{\theta}{3}} \qquad \alpha < \theta < \alpha + 2\pi$$

Ejemplos de determinación de rama

$\alpha = -\pi$





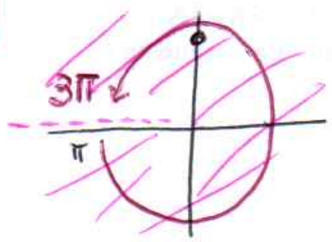
$$-\pi < \theta = \arg z < \pi \longrightarrow -\frac{\pi}{3} < \arg w < \pi/3$$

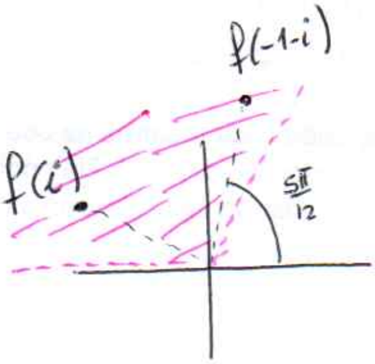
$$f(i) = \sqrt[3]{1}\cdot e^{i\frac{\pi}{2}\cdot\frac{1}{3}} = e^{i\pi/6}$$

$$f(-1-i) = \sqrt[3]{\sqrt{2}}\cdot e^{i\frac{1}{3}\left(-\frac{3\pi}{4}\right)} = \sqrt[6]{2}\, e^{-i\frac{\pi}{4}}$$

Otra rama de raíz cúbica:

$\alpha = \pi$





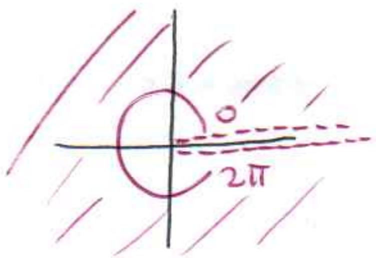
$$\pi < \theta = \arg z < 3\pi \longrightarrow \frac{\pi}{3} < \arg w < \pi$$

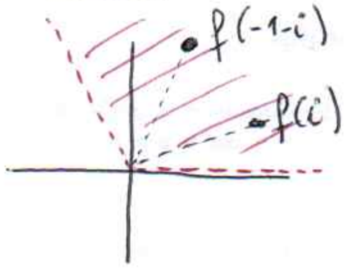
$$f(i) = \sqrt[3]{1}\, e^{i\frac{1}{3}\cdot\frac{5\pi}{2}} = e^{i\frac{5\pi}{6}}$$

$$f(-1-i) = \sqrt[3]{\sqrt{2}}\, e^{i\frac{1}{3}\frac{5\pi}{4}} = \sqrt[6]{2}\, e^{i\frac{5}{12}\pi}$$

Otra determinación con otro corte de rama

$\alpha = 0$





$0 < \theta = \arg z < 2\pi \longrightarrow 0 < \arg w < \frac{2\pi}{3}$

$f(i) = \sqrt[3]{1}\, e^{i\frac{1}{3}\cdot\frac{\pi}{2}} = e^{i\pi/6}$

$f(-1-i) = \sqrt[6]{2}\, e^{i\frac{1}{3}\frac{5\pi}{4}} = \sqrt[6]{2}\, e^{i\frac{5}{12}\pi}$

Estudiar puntos de ramificación y corte de rama de la función $\boxed{g(z) = (z-i)^{1/3}}$ donde la raíz cúbica es la determinada ~~para~~ anteriormente

(Tipo prob 37,38)

g: composición de traslación y raíz cúbica

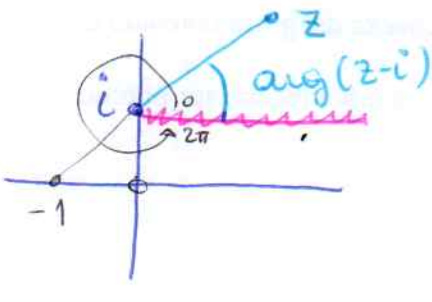
$\downarrow$ $z-i$ holo en $\mathbb{C}$

$\downarrow$ holo en $\mathbb{C}$ − $\underbrace{\text{corte de rama}}_{\text{semieje real +}}$

Observe $\{\ \}^{1/3} = |\{\ \}|^{1/3} \cdot e^{i\frac{1}{3}\arg(\{\ \})}$ → es holo en ptos tales que $\{\ \}$ no es real positivo

$\Rightarrow g(z) = (z-i)^{1/3}$ es holo en $\mathbb{C} - \underbrace{\{z : z-i = x+i0,\ x \geq 0\}}_{\text{corte de rama de } g.}$

Pto de ramif: $z : z-i = 0 \iff \boxed{z = i}$



Algunos valores:

$g(0) = (0-i)^{1/3} = |-i|^{1/3} \cdot e^{i\frac{1}{3}\arg(-i)}$

$= e^{i\pi/2}$

$g(-1) = (-1-i)^{1/3} = \sqrt[3]{\sqrt{2}}\, e^{i\frac{1}{3}\arg(-1-i)}$

$= \sqrt[6]{2}\, e^{i\frac{5}{12}\pi}$

FUNCIONES TRIGONOMETRICAS

Recordemos: $e^{i\theta} = \cos(\theta) + i\,\text{sen}(\theta)$

$$e^{-i\theta} = \cos(-\theta) + i\,\text{sen}(-\theta)$$

$$= \cos(\theta) - i\,\text{sen}(\theta) = \overline{e^{i\theta}}$$

$$e^{i\theta} + e^{-i\theta} = 2\cos(\theta)$$

$$e^{i\theta} - e^{-i\theta} = 2i\,\text{sen}(\theta)$$

Entonces:

$$\boxed{\text{sen}\, z = \frac{e^{iz} - e^{-iz}}{2i}}$$

$$\boxed{\cos z = \frac{e^{iz} + e^{-iz}}{2}}$$

Holomorfas? → sí, en $\mathbb{C}$.

$$\text{sen}' z = \frac{i\, e^{iz} - (-i) e^{-iz}}{2i} = i\left(\frac{e^{iz} + e^{-iz}}{2i}\right) = \cos z$$

$$\cos' z = \frac{i e^{iz} + (-i) e^{-iz}}{2} = i\left(\frac{e^{iz} - e^{-iz}}{2}\right) = -1\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = -\text{sen}\, z$$

Identidades

- $\text{sen}(-z) = -\text{sen}(z)$: $\text{sen}(-z) = \dfrac{e^{i(-z)} - e^{-i(-z)}}{2i} = -\left(\dfrac{-e^{-iz} + e^{iz}}{2i}\right) = -\text{sen}(z)$
- $\cos(-z) = \cos(z)$: $\cos(-z) = \dfrac{e^{i(-z)} + e^{-i(-z)}}{2} = \dfrac{e^{-iz} + e^{iz}}{2} = \cos(z)$
- $\text{sen}(z+2\pi) = \text{sen}(z)$: $\text{sen}(z+2\pi) = \dfrac{e^{i(z+2\pi)} - e^{-i(z+2\pi)}}{2i} =$

$$= \frac{e^{iz} \cdot e^{i2\pi} - e^{-iz} \cdot e^{-i2\pi}}{2i} = \frac{e^{iz} - e^{-iz}}{2i} = \text{sen}\, z$$

- $\cos(z+2\pi) = \cos(z)$

- $\operatorname{sen}(z+\pi) = -\operatorname{sen}(z)$: $\operatorname{sen}(z+\pi) = \dfrac{e^{i(z+\pi)} - e^{-i(z+\pi)}}{2i} = \dfrac{e^{iz}\cdot \overset{-1}{e^{i\pi}} - e^{-iz}\cdot \overset{-1}{e^{-i\pi}}}{2i}$

$$= \frac{-e^{iz}+e^{-iz}}{2i} = -\operatorname{sen}(z)$$

- $\cos(z+\pi) = -\cos(z)$

- $|\operatorname{sen} z|^2 = \left(\dfrac{e^{iz}-e^{-iz}}{2i}\right)\left(\overline{\dfrac{e^{iz}-e^{-iz}}{2i}}\right) = \left(\dfrac{e^{iz}-e^{-iz}}{2i}\right)\left(\dfrac{\overline{e^{iz}}-\overline{e^{-iz}}}{\overline{2i}}\right) =$

$$= \frac{1}{4}\left(e^{iz}\cdot\overline{e^{iz}} - e^{iz}\cdot\overline{e^{-iz}} - e^{-iz}\cdot\overline{e^{iz}} + e^{-iz}\overline{e^{-iz}}\right) =$$

$$= \frac{1}{4}\left(|e^{iz}|^2 - \left(e^{iz}\overline{e^{-iz}} + \overline{e^{iz}\overline{e^{-iz}}}\right) + |e^{-iz}|^2\right) =$$

$$= \frac{1}{4}\left(e^{-2y} - 2\operatorname{Re}\left(e^{iz}\overline{e^{-iz}}\right) + e^{2y}\right) =$$

$$= \frac{1}{4}\left(e^{-2y} - 2\left(\cos^2 x - \operatorname{sen}^2 x\right) + e^{2y}\right) =$$

$$= \frac{1}{4}\left(e^{-2y} - 2\left(\cos^2 x + \operatorname{sen}^2 x - 2\operatorname{sen}^2 x\right) + e^{2y}\right) =$$

$$= \frac{1}{4}\left(e^{-2y} + e^{2y} - 2\right) + \operatorname{sen}^2 x \longrightarrow \operatorname{senh} y = \frac{e^{y}-e^{-y}}{2}$$

- $\boxed{|\operatorname{sen} z|^2 = \operatorname{senh}^2 y + \operatorname{sen}^2 x} \Rightarrow \operatorname{sen}(z)$ NO está acotado! en $\mathbb{C}$

- $\boxed{|\cos z|^2 = \operatorname{senh}^2 y + \cos^2 x} \Rightarrow \cos(z)$ NO está acotado!

- $\operatorname{sen} z = 0 \iff z = n\pi,\ n \in \mathbb{Z}$ $\rightsquigarrow$ $\dfrac{e^{iz}-e^{-iz}}{2i} = 0 \iff e^{iz} = e^{-iz} \iff$

$\log(e^{iz}) = \log(e^{-iz}) \iff$

$iz = -iz + 2n\pi i \quad n \in \mathbb{Z}$

$2z = 2n\pi \iff \boxed{z = n\pi}$

- $\cos z = 0 \iff z = \pi/2 + n\pi,\ n \in \mathbb{Z}$

- $\operatorname{sen}^2 z + \cos^2 z \doteq 1$

$\mathrm{Re}(\mathrm{sen}(z))\ ,\ \mathrm{Im}(\mathrm{sen}(z))\ ?$

$$\mathrm{sen}(z) = \frac{e^{iz}-e^{-iz}}{2i} = \frac{e^{-y}(\cos(x)+i\,\mathrm{sen}(x)) - e^{y}(\cos(x)-i\,\mathrm{sen}(x))}{2i} =$$

$$= \mathrm{sen}(x)\,\frac{i}{i}\left(\frac{e^{-y}+e^{y}}{2}\right) + \frac{\cos(x)}{i}\left(\frac{e^{-y}-e^{y}}{2}\right) =$$

$$= \underbrace{\mathrm{sen}(x)\cosh(y)}_{\mathrm{Re}(\mathrm{sen}(z))} + i\,\underbrace{\cos(x)\,\mathrm{senh}(y)}_{\mathrm{Im}(\mathrm{sen}(z))}$$

$\mathrm{Re}(\cos(z)),\ \mathrm{Im}(\cos(z))\ ?$

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2} = \frac{e^{-y}(\cos(x)+i\,\mathrm{sen}(x)) + e^{y}(\cos(x)-i\,\mathrm{sen}(x))}{2}$$

$$= \cos(x)\left(\frac{e^{-y}+e^{y}}{2}\right) + i\,\mathrm{sen}(x)\left(\frac{e^{-y}-e^{y}}{2}\right) =$$

$$= \underbrace{\cos(x)\cosh(y)}_{\mathrm{Re}(\cos(z))} - i\,\underbrace{\mathrm{sen}(x)\,\mathrm{senh}(y)}_{\mathrm{Im}(\cos(z))}$$

FUNCIONES HIPERBOLICAS

$$\boxed{\operatorname{senh}(z) = \frac{e^{z} - e^{-z}}{2} =}$$

$\downarrow$

$sh(z)$

$$\boxed{\cosh(z) = \frac{e^{z} + e^{-z}}{2}}$$

$\downarrow$

$ch(z)$

* Son enteros : $\operatorname{senh}'(z) = \cosh(z)$

  $\cosh'(z) = \operatorname{senh}(z)$

* $\operatorname{senh}(iz) = i\operatorname{sen}(z)$

  $\cosh(iz) = \cos z$

* $\operatorname{senh}(-z) = -\operatorname{senh}(z)$

  $\cosh(-z) = \cosh(z)$

* $\cosh^2(z) - \operatorname{senh}^2(z) = 1$

* $\operatorname{senh}(z) = \operatorname{senh} x \cos y + i \cosh x \operatorname{sen} y$

  $\cosh(z) = \cosh x \cdot \cos y + i \operatorname{senh} x \operatorname{sen} y$

* $\operatorname{senh}(z + 2\pi i) = \operatorname{senh}(z)$

  $\cosh(z + 2\pi i) = \cosh(z)$